Skew Brownian Motion: A Model for Diffusion with Interfaces?

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Abstract. Skew Brownian motion is a diffusion process on the real line with a distinguished point. A particle diffusing under skew Brownian motion behaves as if it were experiencing ordinary Brownian motion except at the distinguished point, but at the distinguished point it moves to the left with a probability P or to the right with probability $1 - P$, with P not equal to 1/2. As such, skew Brownian motion may be a reasonable model for diffusion with interfaces. An application is given to an ecological model with two types of habitat. However, if skew Brownian motion is formulated in terms of diffusion equations, assuming conservation of mass leads to predictions that in some cases a nonzero number of individuals must be at the distinguished point; i.e. that the probability distribution for the position of a particle may include a delta function at the distinguished point. In the ecological context there is some empirical work suggesting that individuals might in fact sometimes congregate on interfaces. It is unclear whether this behavior is problematic in other modeling contexts. The process of skew Brownian motion should be studied further to assess its usefulness in modeling diffusion in the presence of interfaces.

§1. Introduction

Diffusion processes based on the concept of Brownian motion are widely used to describe biological and chemical phenomena in which different species disperse at random and interact. A variation on standard Brownian motion, known as skew Brownian motion, allows for random dispersal but with an interface at which there is a preferred direction of motion. It is the purpose of this article to give a brief description of skew Brownian motion with some references and a description of how we used it to model a problem in population dynamics related to refuge design.

The idea of skew Brownian motion was introduced as an exercise in a book by Ito and McKean [3, §4.2, problem 1]. The properties of the process were explored to some extent in (Walsh [6], Harrison and Shepp [2]). Essentially, skew Brownian motion describes the position of an object moving on the real line via an unbiased random walk, *except* when it reaches the point $x = 0$,

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where it moves one direction with probability α or the other with probability $1-\alpha$, $\alpha \neq 1/2$. This last property suggests that skew Brownian motion might be a reasonable starting point for the construction of models for dispersal in the presence of interfaces. In fact, Walsh [6] suggests that it might be used to model "a particle diffusing through a semi-permeable membrane under osmotic pressure". However, if the dispersal process is not allowed to increase or decrease the total amount of dispersing material (or number of dispersing individuals) it turns out to be necessary to allow particles (individuals) to actually remain on or in the interface itself, at least temporarily. This point is not mentioned in any of the references on skew Brownian motion, so we perform the derivation in the next section, along with giving a description and formulation of skew Brownian motion. In the third section we discuss a model based on skew Brownian motion which was developed in (Cantrell and Cosner [1]) to describe population dynamics and dispersal in an environment consisting of different types of habitat when the individuals in the population prefer one habitat type over the other. The main point of that discussion is to provide an illustration of how skew Brownian motion can be used in a model. In the last section we draw a few conclusions about the use of skew Brownian motion in modeling and suggest some directions for further research.

§2. Description and Anomalies of Skew Brownian Motion

Ordinary Brownian motion can be described as the limit of a random walk where at each time step $\triangle t$ the corresponding spatial step is either $\triangle x$ or $-\triangle x$, each with probability 1/2, where the limit is taken so that $\triangle x$ and $\triangle t$ approach zero with $(\triangle x)^2/\triangle t = \delta^2$ for some fixed δ. Skew Brownian motion can be constructed in a completely analogous way, *except* that there is a distinguished point (say $x = 0$) for which the probability of moving to the right is α and the probability of moving to the left is $1-\alpha$, with $\alpha \neq 1$ in general. (Harrison and Shepp [2].) Both ordinary and skew Brownian motions are diffusion processes which can be associated with semigroups of operators $\{T_t : t \geq 0\}$ via the Chapman-Kolmogorov equations for their transition functions (see [5]). The infinitesimal generator of the semigroup is computed as

$$Af = \lim_{t \downarrow 0} \frac{T_t f - f}{t}$$

and the associated diffusion equation is

$$\frac{du}{dt} = Au. \tag{1}$$

In the case of ordinary Brownian motion the generator is $A = (\delta^2/2)d^2/dx^2$ with domain effectively being $C^2(\mathbb{R})$ ([5, p.18]) so that the diffusion equation (1) can be realized as

$$\frac{\partial u}{\partial t} = \frac{\delta^2}{2}\frac{\partial^2 u}{\partial x^2} \qquad \text{for } (x,t) \in \mathbb{R} \times (0,\infty). \tag{2}$$

(The transition function itself can be viewed as the fundamental solution of (2).) It is common practice in mathematical modeling to begin with equation (2) as a description of transport or dispersal via diffusion and then to add additional "reaction" terms to describe chemical reactions, population dynamics, etc., and usually this causes no problems. However, some care must be taken in using that approach for skew Brownian motion if we want a transport mechanism which does not change the total amount of mass, population, or whatever quantity is being described by the model. The corresponding semigroup generator for skew Brownian motion was computed by Walsh [6].

$$\begin{aligned} &Af = \frac{\delta^2}{2}\frac{d^2 f}{dx^2}, \\ &\mathrm{dom}A = \{f \in C^2(\mathbb{R}\backslash\{0\}) : \alpha f'(0+) = (1-\alpha)f'(0-), f''(0+) = f''(0-)\}. \end{aligned} \tag{3}$$

The realization of (1) as a differential equation becomes

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\delta^2}{2}\frac{\partial^2 u}{\partial x^2}, \quad (x,t) \in (\mathbb{R}\backslash\{0\}) \times (0,\infty), \\ &\frac{\alpha \partial u}{\partial x}(0+,t) = (1-\alpha)\frac{\partial u}{\partial x}(0-,t), \\ &\frac{\partial^2 u}{\partial x^2}(0+,t) = \frac{\partial^2 u}{\partial x^2}(0-,t). \end{aligned} \tag{4}$$

If (4) is interpreted in terms of fluxes via Fick's law, the flux $(\delta^2/2)\partial u/\partial x$ will be discontinuous at $x = 0$ if $\alpha \neq 1/2$ and $\partial u/\partial x \neq 0$ at $x = 0+$ or $x = 0-$. This suggests that one should expect material to build up on the interface at $x = 0$, which suggests that there might be a nonzero probability that a diffusing particle will be precisely at the single point $x = 0$, which would be inconsistent with the probability measure for the location of the particle being absolutely continuous with respect to Lebsegue measure. To explore these ideas more carefully we restrict (4) to the interval $[-1,1]$ and impose reflecting (i.e. no flux) boundary conditions at $x = \pm 1$. The resulting equation is

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\delta^2}{2}\frac{\partial^2 u}{\partial x^2} \quad \text{on} \quad [(-1,0)\cup(0,1)] \times (0,\infty), \\ &\frac{\partial u}{\partial x}(\pm 1,t) = 0, \quad \alpha\frac{\partial u}{\partial x}(0+,t) = (1-\alpha)\frac{\partial u}{\partial x}(0-,t), \\ &\frac{\partial^2 u}{\partial x^2}(0+,t) = \frac{\partial^2 u}{\partial x^2}(0-,t). \end{aligned} \tag{5}$$

If we think of $u(x,t)$ as a function and integrate (5) over the inverval $[-1,1]$ with respect to x, we obtain

$$\begin{aligned} \frac{d}{dt}\int_{-1}^{1} u(x,t)dx &= \frac{\delta^2}{2}\int_{-1}^{0}\frac{\partial^2 u}{\partial x^2}(x,t)dx + \frac{\delta^2}{2}\int_{0}^{1}\frac{\partial^2 u}{\partial x^2}(x,t)dx \\ &= \frac{\delta^2}{2}\left[\frac{\partial u}{\partial x}(0-,t) - \frac{\partial u}{\partial x}(0+,t)\right] \\ &= \frac{\delta^2}{2}\left(\frac{2\alpha-1}{1-\alpha}\right)\frac{\partial u}{\partial x}(0+,t). \end{aligned} \tag{6}$$

To obtain a model where the total mass is conserved we would need to augment the continuous distribution u determined by (5) with a point mass or Dirac delta at $x = 0$ multiplied by a time dependent coefficient $p(t)$ satisfying

$$\frac{dp}{dt} = -\frac{\delta^2}{2}\left(\frac{2\alpha - 1}{1-\alpha}\right)\frac{\partial u}{\partial x}(0+, t). \tag{7}$$

Since $u(x,t)$ can be determined from (5) (e.g. via the sort of methods used in [1]), equation (7) together with initial data for u and for the total mass at $t = 0$ will determine $p(t)$. The density will then be $u(x,t) + p(t)\delta(x)$, with $p(t) \neq 0$ in general.

§3. An Application

In [1] the goal was to model a situation where a population inhabits a region consisting of two different types of habitat, diffuses freely within each habitat type, but has a preferred direction (i.e. preferred choice of habitat) at the interface between habitat types. We used skew Brownian motion as the paradigm for describing movement at an interface with a preferred direction. The specific scenario we examined was that of a refuge consisting of favorable habitat surrounded by a buffer zone of somewhat unfavorable habitat, with the exterior of the buffer zone assumed to be immediately lethal to the population. The model assumed a local population growth rate r inside the refuge and a death rate $-s$ in the buffer zone. The refuge was taken to be the interval $(0, 2L)$ and the buffer zone to be $(-\ell, 0) \cup (2L, 2L + \ell)$. The model for the population density $u(x,t)$ was then given as

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_2\frac{\partial^2 u}{\partial x^2} + ru \quad \text{in } (0, 2L) \times (0, \infty), \\ \frac{\partial u}{\partial t} &= D_1\frac{\partial^2 u}{\partial x^2} - su \quad \text{in } [(-\ell, 0) \cup (2L, 2L+\ell)] \times (0, \infty), \end{aligned} \tag{8}$$

$$\begin{aligned} \alpha D_1\frac{\partial u}{\partial x}(0+, t) &= (1-\alpha)D_2\frac{\partial u}{\partial x}(0-, t), \\ \alpha D_1\frac{\partial u}{\partial x}(2L-, t) &= (1-\alpha)D_2\frac{\partial u}{\partial x}(2L+, t), \end{aligned} \tag{9}$$

$$\begin{aligned} D_2\frac{\partial^2 u}{\partial x^2} + ru \,|_{x=0+} &= D_1\frac{\partial^2 u}{\partial x^2} - su \,|_{x=0-}\,, \\ D_2\frac{\partial^2 u}{\partial x^2} + ru \,|_{x=2L-} &= D_1\frac{\partial^2 u}{\partial x^2} - su \,|_{x=2L+}\,, \end{aligned} \tag{10}$$

$$u(-\ell, t) = u(2L+\ell, t) = 0. \tag{11}$$

The equations (8) correspond to standard diffusion and growth or decline within each subregion. The interface condition (9) is based on the condition for skew Brownian motion, modified to account for the different diffusion rates in the two regions, with α again representing the probability that an individual on the interface will move into the refuge. If $\alpha = 1/2$ the condition is equivalent to matching fluxes across the interface. For $\alpha > 1/2$ there is a preference for moving into the refuge. The equation (10) arises from the requirement that the domain of the generator A of the diffusion processes should be restricted to functions $f(x)$ on $(-\ell, 2L+\ell)$ such that $(Af)(x)$ is continuous. (This requirement is imposed so that individuals never get permanently stuck at any location and so there are no impenetrable barriers to dispersal; see [3, p.83–100].) Finally, the boundary condition (11) simply means that individuals reaching the boundary die immediately so the density there is zero.

The model (8)-(11) can be analyzed via separation of variables. In [1] the main goal was to determine the dependence of the average population growth rate on the parameters r, s, ℓ, L, and α. The average population growth rate is determined as the principal eigenvalue of the operator A for which (8)-(11) give the realization of $du/dt = Au$. The corresponding eigenvalue problem for A is given by the following, which arise from (8), (10), and (11):

$$\begin{aligned} D_2\frac{\partial^2\phi}{dx^2} + r\phi = \sigma\phi \qquad &\text{in } (0,2L), \\ D_1\frac{\partial^2\phi}{\partial x^2} - s\phi = \sigma\phi \qquad &\text{in } (-\ell,0)\cup(2L,2L+\ell), \end{aligned} \tag{12}$$

$$\begin{aligned} \alpha D_1\frac{\partial\phi}{\partial x}\Big|_{x=0+} &= (1-\alpha)D_2\frac{\partial\phi}{\partial x}\Big|_{x=0-}, \\ \alpha D_1\frac{\partial\phi}{\partial x}\Big|_{x=2L-} &= (1-\alpha)D_2\frac{\partial\phi}{\partial x}\Big|_{x=2L+}, \end{aligned} \tag{13}$$

$$\phi(-\ell) = \phi(2L+\ell) = 0. \tag{14}$$

In view of (12), equation (10) becomes the requirement that $\phi(x)$ must be continuous at $x = 0, 2L$. It is possible to solve (12)-(14) by constructing the eigenfunction ϕ in terms of trigonometric and (possibly) hyperbolic functions and then using the matching conditions at $x = 0$ and $x = 2L$ to determine σ via a transcendental equation. The conclusions of [1] were then obtained by studying the dependence of σ on the parameters r, s, α, ℓ, L. A typical sort of conclusion is that the average growth rate is most sensitive to an increase of the size ℓ of the buffer zone when both the buffer zone and the refuge are small, but the sensitivity to the size of ℓ decreases rapidly as ℓ gets larger ([1, section 4]). Many other results on parameter dependence are also obtained. The key point for the current discussion, however, is simply that a model based on skew Brownian motion proved to be tractable and yielded reasonable and useful conclusions in an applied context.

As is shown by the analysis in [1] it is not always necessary for the mathematical analysis to account for the possibility that there may be a nonzero number of individuals actually *on* the interfaces at any given time. However, this is a point that should be considered when evaluating the model from an applied viewpoint. There is some empirical evidence that when confronted by a barrier dispersing individuals of some species may stop or move along the barrier rather than crossing it or reversing direction [4]. For such species a prediction that some individuals may remain at the interface for some time is at least plausible. However, such an assumption might not be plausible in other contexts, so the modelling approach should be used with some care.

§4. Conclusions

The concept of skew Brownian motion seems to have some potential applications in biological modelling because it can lead to reasonably tractable models which allow for a preferred direction of movement at an interface. However, skew Brownian motion does not appear to have been widely studied from the mathematical viewpoint, so if it is used in a model some care must be taken in the analysis and interpretation of the model. The observation that skew Brownian motion seems to require the introduction of a point mass at the origin to avoid a gain or loss of particles via dispersal (i.e. without reaction dynamics) may be problematic in some applications, although not in others. In any case this point requires further study. Another issue which deserves some attention is that of formulating skew Brownian motion in more than one space dimension. Some other possible directions for further research are described in [1]. We hope that the present discussion will interest some readers enough that they will examine skew Brownian motion themselves and draw their own conclusions.

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